

Geodesic-invariant equations of gravitation

L. Verozub*

Kharkov National University, 61077, Kharkov, Ukraine

Received XXXX, revised XXXX, accepted XXXX

Published online XXXX

Key words space-time, gravitation, equations of gravitation.

PACS 04.20.Cv, 04.50.+h

Einstein's equations of gravitation are not invariant under geodesic mappings, i. e. under a certain class of mappings of the Christoffel symbols and the metric tensor which leave the geodesic equations in a given coordinate system invariant. A theory in which geodesic mappings play the role of gauge transformations is considered.

Copyright line will be provided by the publisher

1 Introduction

If we start with Einstein's beautiful hypothesis that test particles in gravitational field move along geodesic lines of some Riemannian spaces V_4 , it is natural to expect that the differential equations for finding the metric tensor $g_{\alpha\beta}(x)$ for a given distribution of matter also should be invariant under any transformations at which the geodesic equations remain invariant. However, the geodesic equations are invariant not only under arbitrary transformation of coordinates (it is rather obvious) but also under geodesic mappings $\Gamma_{\beta\gamma}^\alpha(x) \rightarrow \bar{\Gamma}_{\beta\gamma}^\alpha(x)$ of the Christoffel symbols in any fixed coordinate system [1]- [4].

If $\Gamma_{\beta\gamma}^\alpha$ are Christoffel symbols in some coordinate system (x^0, x^1, x^2, x^3) , and we use coordinate time $t = x^0/c$ (c is speed of light) as a parameter along geodesic lines, then the differential equations of a geodesic line are of the form

$$\ddot{x}^\alpha + (\Gamma_{\beta\gamma}^\alpha - c^{-1}\Gamma_{\beta\gamma}^0 \dot{x}^\alpha) \dot{x}^\beta \dot{x}^\gamma = 0, \quad (1)$$

where $\dot{x}^\alpha = dx^\alpha/dt$, $\ddot{x}^\alpha = d\dot{x}^\alpha/dt$. It easily to verify that these equations are invariant under the mapping

$$\bar{\Gamma}_{\beta\gamma}^\alpha(x) = \Gamma_{\beta\gamma}^\alpha(x) + \delta_\beta^\alpha \phi_\gamma(x) + \delta_\gamma^\alpha \phi_\beta(x), \quad (2)$$

where $\phi_\alpha(x)$ is a continuously differentiable vector field. In Riemannian space-time $\phi_\alpha(x)$ can be expressed in terms of determinants g and \bar{g} of the metric tensors before and after the geodesic mapping of space-time V into \bar{V} as follows:

$$\phi_\alpha = \frac{1}{n+1} \left(\Gamma_{\alpha\gamma}^\gamma - \bar{\Gamma}_{\alpha\gamma}^\gamma \right) = \frac{1}{2(n+1)} \frac{\partial}{\partial x^\alpha} \ln \left| \frac{\bar{g}}{g} \right|. \quad (3)$$

Such mappings of the Christoffel symbols in a given coordinate system induce some transformations not only of the curvature and Ricci tensors but also transformations $g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta}$ of the metric tensor which are obtained by solving of the partial differential equation

$$\bar{g}_{\beta\gamma;\alpha}(x) = 2\phi_\alpha(x)\bar{g}_{\beta\gamma}(x) + \phi_\beta(x)\bar{g}_{\gamma\alpha}(x) + \phi_\gamma(x)\bar{g}_{\alpha\beta}(x), \quad (4)$$

* Corresponding author: e-mail: l.verozub@googlemail.com,

where the semicolon denotes a covariant derivative with respect to the metric in V_4 .

It is clearly that all solutions of these equation are, in principle, equivalent physically. However Einstein's equations even in vacuum are not invariant with respect to geodesic mappings [5].

It follows from (2) that the components $\bar{\Gamma}_{00}^i = \Gamma_{00}^i$. Therefore, in Newtonian limit geodesic-invariance is not an essential fact. However this cannot be said about the relativistic case. For this reason is of interest to explore a theory in which the gravitation equations, as well as the equations of motion of test particles, are geodesic-invariant i.e. in which geodesic invariance plays the role of gauge invariance.

In Sect. 2 a geodesic-invariant generalization of Einstein's equations is considered. It turns out that the simplest equations of such a kind are some bimetric equations. In Sects 3 to 7, proceeding from new sight at Poincaré old fundamental ideas, we argue that the both spaces, Riemannian (with the curvature other than zero) and at, can have physical sense depending on which kind of reference frame is used. In Sects. 8, 9 we find the spherically-symmetric solution of the above equations. Finally, in Sect. 10 it is shown that features of the gravitational force resulting from this solution, well explain properties of the Hubble diagram obtained from the latest observations.

2 Generalization of Einstein's equations

The simplest way to find a gauge-invariant generalization of Einstein's equations is to consider components $\Gamma_{\alpha\beta}^\gamma$ of the affine connection in $n = 4$ -dimensional manifold \mathcal{M}_4 as 4-components of some affine connection Γ_{BC}^A of $n + 1$ dimensionall manifold \mathcal{M}_5 in which transformations (2) are consequences of 5th coordinate transformations [2]:

$$\bar{x}^4 = x^4 - \int_0^x \phi_\beta(x') dx'^\beta, \quad (5)$$

where $\phi_\beta(x')$ is a gradient vector field. These transformations together with 4-transformations

$$\bar{x}^\alpha = \bar{x}^\alpha(x^0, x^1, x^2, x^3) \quad (6)$$

form some acceptable coordinate transformations in \mathcal{M}_5 . In \mathcal{M}_5 one can define an affine connection by the functions Γ_{AB}^A which are transformed under (5) and (6) as follows:

$$\bar{\Gamma}_{BC}^A = \left(\Gamma_{ED}^F \frac{\partial x^E}{\partial \bar{x}^B} \frac{\partial x^D}{\partial \bar{x}^C} + \frac{\partial^2 x^F}{\partial \bar{x}^B \partial \bar{x}^C} \right) \frac{\partial \bar{x}^A}{\partial x^F}. \quad (7)$$

The equations for 4-components:

$$\bar{\Gamma}_{\beta\gamma}^\alpha = \left(\Gamma_{EC}^F \frac{\partial \bar{x}^E}{\partial x^\beta} \frac{\partial x^C}{\partial \bar{x}^\gamma} + \frac{\partial^2 x^F}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \right) \frac{\partial \bar{x}^\alpha}{\partial x^F} \quad (8)$$

under transformations (5) takes the form (2) if Christoffel's symbols satisfy the conditions : $\Gamma_{\gamma 4}^\alpha = \Gamma_{4\gamma}^\alpha = \delta_\gamma^\alpha$ and $\Gamma_{44}^\alpha = 0$. These conditions are invariant under coordinate transformations (6) and (5). The rest components of Γ_{BC}^A should satisfy the conditions which also are obtained from consideration of their transformation properties under (5). The components $\bar{\Gamma}_{0\gamma}^4 = 0$ in (7) do not depend on Γ_{ED}^F . This allows to set $\Gamma_{0\gamma}^4 = 0$ in any coordinate system. The component $\bar{\Gamma}_{44}^4$ are equal to Γ_{44}^4 , which allows to set $\Gamma_{44}^4 = 1$ in any coordinate system. The components $\Gamma_{\alpha\beta}^4$ are transformed as a tensor under coordinate transformations (6), and as

$$\bar{\Gamma}_{\alpha\beta}^4 = \Gamma_{\alpha\beta}^4 - \psi_{\alpha\beta} \quad (9)$$

under transformations (5) of the 5th coordinate where $\psi_{\alpha\beta} = \psi_{\alpha;\beta} - \psi_\alpha \psi_\beta$, and $\psi_{\alpha;\beta}$ is a covariant derivative in V_4 .

The object

$$\mathcal{R}_{BCD}^A = \partial \Gamma_{BC}^A / \partial x^D \partial \Gamma_{BD}^A / \partial x^C + \Gamma_{MD}^A \Gamma_{BC}^M - \Gamma_{MC}^A \Gamma_{BD}^M \quad (10)$$

is a curvature tensor in \mathcal{M}_5 . In our case the components $\mathcal{R}_{4\alpha} = \mathcal{R}_{\alpha 4}$ and the component \mathcal{R}_{44} of the contracted tensor $\mathcal{R}_{BC} = \mathcal{R}_{BAC}^A$ are equal to zero identically. For this reason $\mathcal{R}_{\beta\gamma}$ is the 4- tensor

$$\mathcal{R}_{\beta\gamma} = R_{\beta\gamma} + (n-1)\Gamma_{\beta\gamma}^4, \quad (11)$$

where $R_{\beta\gamma}$ is the Ricci tensor.

The components of the Ricci tensor are transformed under (5) as follows:

$$\bar{R}_{\alpha\beta} = R_{\alpha\beta} - \psi_{\alpha\beta}. \quad (12)$$

On account of (9) the object $\mathcal{R}_{\beta\gamma}$ is gauge (i.e geodesic) invariant. For this reason, the simplest geodesic-invariant generalization of vacuum Einstein's equations is of the form

$$R_{\beta\gamma} + (n-1)\Gamma_{\beta\gamma}^4 = 0, \quad (13)$$

where $\Gamma_{\beta\gamma}^4$ is a tensor field which is transformed under geodesic mappings according to (9).

Now the problem is to find some relevant tensor $\Gamma_{\beta\gamma}^4$. Such object must satisfy the following requirements:

1. It should transform under (5) according to (9),
2. It should transform under (6) as a 4-tensor,
3. It should lead to such generalization of Einstein's equations which solutions can noticeably differ from the solutions of the last equations only in strong gravitation field as in weak and moderately strong field solutions of Einstein equations are in good agreement with observations.

The first requirement means that $\Gamma_{\beta\gamma}^4$ should be formed from geometrical objects of space-time V_4 , and cannot be an additional field. Because the equations of motion of free particles contains connection coefficients (and not metric) it should be formed from components of $\Gamma_{\beta\gamma}^\alpha$.

The simplest object $\Gamma_{\beta\gamma}^4$ which can be formed by components of Christoffel symbols and which has the requirement transformations properties under our gauge transformations is

$$\Gamma_{\alpha\beta}^4 = \frac{1}{n+1}(\Gamma_{\alpha,\beta} - \Gamma_{\alpha\beta}^\gamma \Gamma_\gamma) - \frac{1}{(n+1)^2} \Gamma_\alpha \Gamma_\beta, \quad (14)$$

where $\Gamma_\alpha = \Gamma_{\alpha\gamma}^\gamma$. In this case $\mathcal{R}_{\beta\gamma}$ is an object which is formed by gauge-invariant Thomas symbols [2]

$$\Pi_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - (n+1)^{-1} \left[\delta_\alpha^\gamma \Gamma_\beta + \delta_\beta^\gamma \Gamma_\alpha \right], \quad (15)$$

the same way as the Ricci tensor is formed out of the Christoffel symbols. As $\Pi_{\alpha\gamma}^\alpha = 0$, equations (13) take the form

$$\nabla_\gamma \Pi_\beta^\alpha - \Pi_{\beta\delta}^\epsilon \Pi_{\epsilon\gamma}^\delta = 0 \quad (16)$$

where ∇_α denotes a covariant derivative in Minkowski space-time.

However, the problem is that $\mathcal{R}_{\beta\gamma}$ is not a tensor with respect to general coordinate transformations because $\Gamma_\alpha(x)$ is not a vector field.

This a new problem evidently cannot be solved within the framework of General Relativity because by using components $\Gamma_{\beta\gamma}^\alpha$ it is impossible to form a tensor of the second rank in V_4 which is invariant under geodesic transformations.

However a satisfactory solution of this problem can be found if the Riemannian space-time V_4 coexists in the manifold \mathcal{M}_4 with Minkowski space-time E_4 which also has a physical meaning. From the point of

view of such bimetric structure of space-time, in Minkowski space-time $g_{\alpha\beta}(x)$ is simply some tensor field. The Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ can be considered as components of the tensor $D_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \overset{\circ}{\Gamma}_{\beta\gamma}^\alpha$ in Cartesian coordinates, where $\overset{\circ}{\Gamma}_{\beta\gamma}^\alpha$ is the Christoffel symbols in Minkowski space-time. Tensor $D_{\beta\gamma}^\alpha$ is formed by substitution of ordinary derivatives in $\Gamma_{\alpha\beta}^\gamma$ by the covariant ones ² because both the definitions of the tensor coincide in Cartesian coordinates. The object Γ_α is components of the vector $D_\alpha = D_{\alpha\beta}^\beta = \Gamma_{\alpha\beta}^\beta - \overset{\circ}{\Gamma}_{\alpha\beta}^\beta$ in Cartesian coordinates.

In such bimetric manifold we can set

$$\Gamma_{\alpha\beta}^4 = Q_{\alpha,\beta} - \Gamma_{\alpha\beta}^\gamma Q_\gamma - Q_\alpha Q_\beta = Q_{\alpha;\beta} - Q_\alpha Q_\beta, \quad (17)$$

where

$$Q_\alpha = \frac{1}{n+1} D_\alpha = \frac{1}{2(n+1)} \frac{\partial}{\partial x^\alpha} \ln \left| \frac{g}{\eta} \right|. \quad (18)$$

The definition (14) is the tensor (17) in Cartesian coordinates.

Similarly to it, instead of Thomas's symbols (15) we have the gauge-invariant tensor

$$B_{\alpha\beta}^\gamma = D_{\alpha\beta}^\gamma - \left[\delta_\alpha^\gamma Q_\beta + \delta_\beta^\gamma Q_\alpha \right] \quad (19)$$

It can be written also as

$$B_{\beta\gamma}^\alpha = \Pi_{\beta\gamma}^\alpha - \overset{\circ}{\Pi}_{\beta\gamma}^\alpha \quad (20)$$

where

$$\overset{\circ}{\Pi}_{\alpha\beta}^\gamma = \overset{\circ}{\Gamma}_{\alpha\beta}^\gamma - (n+1)^{-1} \left[\delta_\alpha^\gamma \overset{\circ}{\Gamma}_\beta + \delta_\beta^\gamma \overset{\circ}{\Gamma}_\alpha \right] \quad (21)$$

are Thomas symbols for E_4 in the used coordinate system.

Therefore, Eqs. (16) can be considered as a particular case of the more general equations

$$\nabla_\alpha B_{\beta\gamma}^\alpha - B_{\beta\delta}^\epsilon B_{\epsilon\gamma}^\delta = 0, \quad (22)$$

where ∇_α denotes a covariant derivative in E_4 .

The above can be generalized to the case of matter presence. It is naturally to consider the tensor $g_{\alpha\beta}$ as 4-components of some metric tensor g_{AB} in the manifold \mathcal{M}_5 :

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} & g_{\alpha 4} \\ g_{4\alpha} & g_{44} \end{pmatrix}. \quad (23)$$

The components $g_{\alpha\beta}$ are transformed under (5) as follows:

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + g_{44}\phi_\alpha\phi_\beta + g_{4\alpha}\phi_\beta + g_{4\beta}\phi_\alpha \quad (24)$$

$$\bar{g}_{\alpha 4} = g_{\alpha 4} + g_{44}\phi_\alpha \quad (25)$$

$$\bar{g}_{44} = g_{44}. \quad (26)$$

Transformations of the components Γ_α under geodesic mappings are given by

$$\bar{\Gamma}_\alpha = \Gamma_\alpha + (n+1)\phi_\alpha, \quad (27)$$

² There is an analogy here with the formalism of the bimetric Rosen theory [33].

which coincides with the transformations of the components $g_{\alpha 4}$ if $g_{44} = n + 1$. For this reason we can assume that³

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} & Q_\alpha \\ Q_\alpha & n+1 \end{pmatrix}. \quad (28)$$

Then there is a gauge-invariant tensor

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta} - (n+1)^{-1} Q_\alpha Q_\beta. \quad (29)$$

Thus, finally, the gauge-invariant generalization of Einstein's equations with matter are of the form

$$\nabla_\alpha B_{\beta\gamma}^\alpha - B_{\beta\delta}^\epsilon B_{\epsilon\gamma}^\delta = k \left(T_{\alpha\beta} - \frac{1}{2} \hat{g}_{\alpha\beta} T \right), \quad (30)$$

where $k = 8\pi\gamma/c^4$, γ is the gravitational constant, c is the speed of light, $T_{\alpha\beta}$ is the matter energy-momentum tensor. At the gauge conditions $Q_\alpha = 0$ equations (30) coincides with Einstein's equations.

Of course, these bimetric equations may be true if and only if both the space-times, V_4 and E_4 , has some physical meaning. How these two physical space-time can coexist? Attempts to use flat space-time simultaneously with curved in general relativity were undertaken repeatedly. Here it should be mentioned the well known Rosen's bimetric theory [6] which leads to non-Einstein equations for $g_{\alpha\beta}(x)$, and interpretations of the Einstein's equations as the equations for $g_{\alpha\beta}$ considered as some function of a tensor field $\psi_{\alpha\beta}$ which describes gravitational field in Minkowski space-time [7], [8], [9], [10]. The need to go beyond Riemannian manifold has noted been also in [11].

It should be noted that at any concrete expressions of $g_{\alpha\beta}$ in terms of $\psi_{\alpha\beta}$, which are used in some papers, the Einstein equations, considered as a function of $\psi_{\alpha\beta}$, cannot be invariant with respect to possible gauge transformations of the field $\psi_{\alpha\beta}$. In the paper [12] it was shown that the requirements of the invariance of the any equations for $g_{\alpha\beta}$ with respect to the possible gauge transformations of $\psi_{\alpha\beta}$ leads uniquely to geodesic invariance of gravitational equation, and the simplest generalization of Einstein's vacuum equations are equations (13) as obtained above.

However bimetric nature of physical space-time belongs probably to one of the deepest problems of physics. In Sec. 3 to 7 we show how both the kind of space-time can coexist in the theory.

3 Relativity of space-time

At the beginning of the 20th century H. Poincaré convincingly showed that it makes no sense to assert that one or other geometry of physical space is true. Only an aggregate "geometry + measuring instruments" has a physical, meaning verifiable by experiment. Einstein recognized that Poincaré reasoning is justified. His theory of gravity does not reject Poincaré's ideas. It only demonstrates relativity of space and time geometry with respect to distribution of matter. Poincaré's ideas indicate to another property of physical reality — relativity of the geometry in relation to measuring instruments, dependency on their properties. These ideas, apparently, have never been realized in physics and, evidently, remain until now a subject for discussion of philosophers. The success of General Relativity almost convinced us that physical space-time in the presence of matter is Riemannian and that this fact does not depend on properties of measuring instruments. Attempts to describe gravity in flat space-time have not obtained recognition and Poincaré's ideas have proved to be almost forgotten for physics.

However, are Poincaré's ideas really only a thing of such little use as conventionalism, as it is usually believed? In fact, a choice of certain properties of measuring instruments is nothing more than the choice of some frame of reference, which is just such a physical device by means of which we test properties of space-time. (Of course, we must understand the distinction between a frame of reference as a physical device, and a coordinate system as only a means of parameterization of events in space-time.) For this reason

³ It means that there is some relationship between Γ_{BC}^A and g_{AB} distinct from such relationship in Riemannian geometry.

Poincaré's reasoning about interdependency between properties of space-time and measuring instruments should be understood as the existence of a connection between geometry of space-time and properties of the employed reference frame .

The existence of such a connection can be shown by an analysis of a simple and well-known example considering it from a new point of view. Disregarding the rotation of the Earth, a reference frame, rigidly connected with the Earth surface, can be considered as an inertial frame (IFR). An observer, located in this frame, can describe the motion of freely falling identical point masses as taking place in Minkowski space-time under the action of a certain force field. However, consider another observer, who is located in a frame of reference, the reference body of which is formed by these freely falling particles. Such a reference frame can be named the proper frame of reference (PFR) for the given force field. Let us assume that the observer is deprived of the possibility of seeing the Earth and stars. This observer does not feel the presence of the force field in any place of his frame. Therefore, if he proceeds from the relativity of space-time in the Berkeley-Leibnitz-Mach-Poincaré (BLMP) sense [13] - [16], then from his viewpoint accelerations of the point masses, forming the reference body of his frame, in his physical space must be equal to zero. However, instead of this, he observes a change in distances between these point masses in time. How can he explain this fact? Evidently, the only reasonable explanation for him is the interpretation of this observed phenomenon as a manifestation of the deviation of geodesic lines in some Riemannian space-time of a nonzero curvature. Thus, if the first observer, located in the IFR, can postulate that space-time is flat, the second observer, located in a PFR of the force field, who proceeds from relativity of space and time in the BLMP sense, already in the Newtonian approximation *is forced* to consider space-time as Riemannian with curvature other than zero.

In next Section it is shown that a similar dual description is possible also for the motion of a perfect isentropic fluid.

4 Space-time metric form in NIFRs

It is possible to arrive at some quantitative relationship between properties of the used frames of reference and the local metric properties of space-time in PRFs examining the problem from more general positions.

Presently we do not know how the space-time geometry in an inertial frame of reference (IFR) is related with the frame properties. Under the circumstances it can simply be postulated, according to Special Relativity, that space-time in IFRs is pseudo-Euclidean. Proceeding from it, one can find the line element of space-time in non-inertial frames of reference (NIFR) from the viewpoint of observers located in the NIFRs and proceeded from relativity of space and time in the BLMP meaning.

By a non-inertial frame of reference we mean the frame, the body of reference of which is formed by point masses moving in an IFR under the effect of a given force field.

The reference body (RB) of a reference frame is supposed to be formed by identical point masses m . If an observer in the reference frame is at rest, his world line coincides with the world line of some point of the reference body. It is obvious for such an observer in an IFR that the accelerations of the point masses forming his reference body are equal to zero. This fact takes place in relativistic meaning, too. That is, if the line element of space-time in the IFR is denoted by $d\sigma$ and $u^\alpha = dx^\alpha/d\sigma$ is the field of 4-velocity of the point masses forming the reference body, then the absolute derivative of u^α is equal to zero: ⁵

$$Du^\alpha/d\sigma = 0. \quad (31)$$

(We mean that an arbitrary coordinate system is used.)

Does it occur for an observer in NIFRs ? That is, if the differential metric form of space-time in a NIFR is denoted by ds , does the 4-velocity vector $\zeta^\alpha = dx^\alpha/ds$ of the point-masses forming the reference body of the NIFR satisfy the equation

$$D\zeta^\alpha/ds = 0 ? \quad (32)$$

⁵ We use notations and definitions, following the Landau and Lifshitz book [22].

If the space-time is absolute, equation (32) holds for only $ds = d\sigma$. However, if space and time are relative in the BLMP sense, then for both observers, located in some IFR and NIFR, the motion of the point masses, forming their reference bodies, which are kinematically equivalent, must be dynamically equivalent, too (both in non-relativistic and relativistic sense). Any observer in the NIFR, isolated from the external world and proceeded from relativity of space-time in BLMP meaning, consider points of the reference body as the ones of his physical space, and space of events as his space-time. Therefore, from his viewpoint point masses forming the reference body of his frame are not under action of any forces (the same as for the observer in IFR), and their 4-velocity must be equal to zero. In other words, since for the observer in the IFR world lines of the reference body are, according to (31), some geodesic lines, for the observer in the NIFR the world lines of the of his BR also must be geodesic lines in his space-time, which can be expressed by (32).

The equation (32) uniquely determines the fundamental metric form in NIFRs. Indeed, the differential equations of these world lines are at the same time Lagrange equations describing in Minkowski space-time the motion of the point masses forming the reference bodies of the NIFR. The last equations can be obtained from a Lagrange action S by the principle of the least action. Therefore, the equations of the geodesic lines can be obtained from a differential metric form $ds = k dS$, where k is constant, $dS = \mathcal{L}(x, \dot{x})dt$, and $\mathcal{L}(x, \dot{x})$ is a Lagrange function describing in Minkowski space-time the motion of identical point masses m forming the body reference of the NIFR. The constant k is equal to $-(mc)^{-1}$, as it follows from the analysis of the case when the frame of reference is inertial.

Thus, if we proceed from relativity of space and time in the BLMP sense, then the line element of space-time in NIFRs can be expected to have the following form

$$ds = -(mc)^{-1} dS(x, dx). \quad (33)$$

Therefore, properties of space-time in NIFRs are entirely determined by properties of used frames in accordance with the BLMP idea of relativity of space and time, which has been noted for the first time in [19].

Consider two examples of NIFRs.

1. The reference body is formed by noninteracting electric charges, moving in a constant homogeneous electric field \mathcal{E} . The motion of the charges in an IFR is described in the Cartesian coordinates system by a Lagrangian [22]

$$L = -mc^2 (1 - v^2/c^2)^{1/2} + \mathcal{E} e x, \quad (34)$$

where v is the speed of a particle. According to (33) the space-time metric differential form in this frame is given by

$$ds = d\sigma - (wx/c^2)dx^0, \quad (35)$$

where

$$d\sigma = [\eta_{\alpha\beta} dx^\alpha dx^\beta]^{1/2}$$

is the differential metric form of the Minkowski space-time in the IFR in the coordinate system being used, and $w = e\mathcal{E}/m$ is the acceleration of the charges.

2. The reference body consists of noninteracting electric charges in a constant homogeneous magnetic field H directed along the axis z . The Lagrangian describing the motion of the particles can be written as follows [22]:

$$L = -mc^2 (1 - v^2/c^2)^{1/2} - (m\Omega_0/2)(\dot{x}y - x\dot{y}), \quad (36)$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, and $\Omega_0 = eH/2mc$.

The points of such a system rotate in the plane xy around the axis z with the angular frequency

$$\omega = \Omega_0 [1 + (\Omega_0 r/c)^2]^{-1/2}, \quad (37)$$

where $r = (x^2 + y^2)^{1/2}$. The linear velocity of the BR points tends to c when $r \rightarrow \infty$.

For the given NIFR

$$ds = d\sigma + (\Omega_0/2c) (ydx - xdy). \quad (38)$$

In the above NIFRs ds is of the form

$$ds = \mathcal{F}(x, dx), \quad (39)$$

where $\mathcal{F}(x, dx) = d\sigma + f_\alpha(x)dx^\alpha$, f_α is a vector field. Therefore, $\mathcal{F}(x, dx)$ is a homogeneous function of the first degree in dx^α , that is $\mathcal{F}(x, ky) = k\mathcal{F}(x, y)$. Thus, the space-time in the above NIFRs is Finslerian [23].

By using the identities

$$\mathcal{F}(x, y) = \frac{\partial \mathcal{F}(x, y)}{\partial y^\alpha} \xi^\alpha; \quad \frac{\partial^2 \mathcal{F}(x, y)}{\partial y^\alpha \partial y^\beta} = 0, \quad (40)$$

the function $\mathcal{F}^2(x, y)$ can be written as

$$\mathcal{F}(x, y)^2 = G_{\alpha\beta}(x, y) y^\alpha y^\beta, \quad (41)$$

where

$$G_{\alpha\beta}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}(x, y)^2}{\partial y^\alpha \partial y^\beta} \quad (42)$$

is an analog of the fundamental metric tensor of Riemannian space-time. Consequently, the modulus of the vector y^α in the point x is $\|y\| = \mathcal{F}(x, y) = [G_{\alpha\beta}(x, y) y^\alpha y^\beta]^{1/2}$.

For ds of the form (39) we have

$$G_{\alpha\beta}(x, y) = \eta_{\alpha\beta} + f_\alpha f_\beta + (\eta_{\alpha\beta} - \tau_\alpha \tau_\beta) f_\sigma \tau^\sigma + f_\alpha \tau_\beta + f_\beta \tau_\alpha, \quad (43)$$

where $\tau^\alpha = dx^\alpha/d\sigma$.

A covariant vector in the Finslerian space-time, therefore, can be defined as

$$y_\alpha^* = \mathcal{F}(x, y) \frac{\partial \mathcal{F}(x, y)}{\partial y^\alpha} = G_{\alpha\beta}(x, y) y^\beta. \quad (44)$$

The orthogonality of vectors y^α and y_1^α can be defined by the equality $y_\alpha^* y_1^\alpha = 0$. It must be noted that the orthogonality of two vectors is not symmetric.

3. The reference body is formed by particles of an isentropic flow of the perfect fluid.

It is well known that besides of traditional continual description, a perfect fluid can be considered as a collection of a finite number of identical macroscopic small particles which are under influence of interparticles forces which mimic the effect of pressure, viscosity etc [17], [18]. In particular, the fluid velocity in a given point is simply velocity of the particle being in this point. At such description the motion of the fluid particles is governed by solutions of ordinary differential equations of Newtonian or relativistic dynamics.

The motion of the fluid particles in an IFR is described by Lagrangian [24]

$$L = -mc \left(G_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2} d\lambda, \quad (45)$$

where λ is a parameter along 4-path of particles, w is enthalpy per unit volume, $G_{\alpha\beta} = \varkappa^2 \eta_{\alpha\beta}$, $\varkappa = w/\rho c^2$, $\rho = mn$, m is the mass of the particles, n is the particles number density, and $\eta_{\alpha\beta}$ is the metric tensor in Minkowski space-time. According to (33) the line element of space-time in this frame is given by

$$ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta. \quad (46)$$

The covariant derivative of tensor $G_{\alpha\beta}$ is equal to zero. Therefore, space-time in such NIFR is Riemannian with the curvature other than zero. The last description is equivalent to the one by the Lagrangian (45) [24].

Thus, an observer in some laboratory (inertial) frame of reference can describe a fluid motion as what is going on in Minkowski space-time by a Lagrangian. However, the observer which is located in a comoving frame of reference cannot feel presence of the force field arising from a pressure gradient since he moves along a geodesic line of some Riemannian space-time. Therefore, if he is isolated from external world, he is forced to explain relative motions of nearby particles of the fluid as a manifestation of deviation of geodesic lines due to space-time curvature in this frame of reference.

5 3+1 decomposition of space-time in NIFRs and Sagnac effect

For the 3 + 1 decomposition of space-time in non-inertial frames of reference into physical 3-space and time we use a covariant method which is a Finslerian generalization of the well-known method in General Relativity [25], [26].

An ideal clock is a local periodic process measuring the length of its own world line on a certain scale. For an observer in a NIFR the direction of physical time in a point x^α is given by the vector of the 4-velocity $\zeta^\alpha = dx^\alpha/ds$ of the given point of the NIFR reference body.

By analogy with the case of Riemannian geometry, an arbitrary 4-vector ξ^α in the point x^α can be represented as follows:

$$\xi^\alpha = \bar{\xi}^\alpha + \beta \zeta^\alpha. \quad (47)$$

In this equation β is a function of x^α , and $\bar{\xi}^\alpha$ is a 3-spatial component of the vector ξ^α . It is supposed to be satisfied to Finslerian's orthogonality condition [23] to the time direction which we define as

$$\zeta_\alpha^* \bar{\xi}^\alpha = 0, \quad (48)$$

where ζ_α^* is the covariant components of the vector ζ^α :

$$\zeta_\alpha^* = \mathcal{F}(x, \zeta) \frac{\partial \mathcal{F}(x, \zeta)}{\partial \zeta^\alpha}. \quad (49)$$

Since $\mathcal{F}(x, \zeta) = 1$, this vector is

$$\zeta_\alpha^* = \frac{\eta_{\alpha\beta} \zeta^\beta}{(\eta_{\alpha\beta} \zeta^\alpha \zeta^\beta)^{1/2}} + f_\alpha. \quad (50)$$

By multiplying (47) with the vector ζ_α^* we find that $\beta = \zeta_\alpha^* \xi^\alpha$ and

$$\bar{\xi}^\alpha = H_\beta^\alpha \xi^\beta, \quad (51)$$

where $H_\beta^\alpha = \zeta^\alpha \zeta_\beta^* - \delta_\beta^\alpha$ is an operator of spatial projection, and δ_β^α is the Kronecker delta.

For the vector $\xi^\alpha = dx^\alpha$ (47) yields

$$dx^\alpha = \bar{dx}^\alpha + c dT_f \zeta^\alpha, \quad (52)$$

where \overline{dx}^α is the spatial components of the vector dx^α , and

$$dT_f = c^{-1} \zeta_\alpha^* dx^\alpha \quad (53)$$

is the time element between events in the points x^α and $x^\alpha + dx^\alpha$ in the NIFR.

Metric form (39) and the spatial projection of the vector dx^α leads to the following covariant form of the spatial element in NIFRs

$$dL_f = (-\eta_{\alpha\beta} \overline{dx}^\alpha \overline{dx}^\beta)^{1/2} + f_\alpha \overline{dx}^\alpha. \quad (54)$$

This covariant equation is the simplest and clearest in the "comoving" coordinate systems, in which

$$\zeta^\alpha = \varepsilon \delta_0^\alpha. \quad (55)$$

It follows from the equality $\mathcal{F}(x, \zeta) = 1$ that $\varepsilon = (\eta_{00}^{1/2} + f_0)^{-1}$.

In this coordinate system for an observer at rest the time element $dT_f = (\eta_{00}^{1/2} + f_0) dx^0$, $\overline{dx}^i = dx^i$, and $\overline{dx}^0 = 0$. Therefore,

$$dL_f = (-\eta_{ik} dx^i dx^k)^{1/2} + f_i dx^i = dL(1 + f_i k^i), \quad (56)$$

where $dL = (-\eta_{ik} dx^i dx^k)^{1/2}$ is the Euclidean spatial element and $k^i = dx^i/dl$ is the unit direction vector.

Of course, the above definitions of dT_f and dL_f have physical meaning only in the space-time region where they are real and positive quantities. In this respect, space-time in NIFRs has more complicated structure as compared with IFRs, and depends on properties of the considered NIFR. For example, space-time in the first NIFR in Sec. 4 is time-like only if the interval $d\sigma$ is time-like and the distance $|x|$ of charges from the coordinate origin satisfy the condition $|x| < c^2/w$.

The Finslerian geometry of space-time in NIFRs allows us to give a natural explanation of the well known Sagnac effect, i.e. of a phase shift in the interference of two oppositely directed coherent light beams on a rotating disk [27]. It is usually, for relativistic explanation of this effect the motion of light in a NIFR is considered as a relative motion in the Pseudo-Euclidean space-time of some IFR [28]. However, for an observer located in a rotating frame and isolated from the IFR (in a "black box"), who proceeds from the notion of space and time relativity in BLMP sense, the observed anisotropy in time of light propagation clockwise and counterwise is not a trivial effect. It must have some "internal" physical explanation.

Consider a rigid disk rotating in the plane xy with a constant angular velocity Ω around the axis z . Let r and θ be the coordinates, defined by the equations

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad \varphi = \theta + \Omega t. \quad (57)$$

In the coordinate system (r, θ, z, t) the differential metrical form ds is of the form

$$ds = d\sigma - (\Omega r^2/2c) d\theta - (\Omega^2 r^2/2c^2) dx^0, \quad (58)$$

where $d\sigma$ is the pseudo - Euclidean metric form:

$$d\sigma^2 = (1 - \Omega^2 r^2/c^2) dx^{0^2} - dr^2 - r^2 d\theta^2 - 2(r^2 \Omega/c) d\theta dx^0 - dz^2. \quad (59)$$

At $\Omega R/c \ll 1$ this reference frame is identical to the NIFR described in example 2 of Sec. 4, and the used coordinate system on the disk is "comoving", where (55) holds.

It follows from (56) that the spatial element in the NIFR is anisotropic. We will show that the speed of light in the noninertial frame of reference is anisotropic, too. Along photon paths the equality $\eta_{\alpha\beta} dx^\alpha dx^\beta = 0$ holds. By using (52), this equation can be written in terms of notions of the NIFR as

$$\eta_{\alpha\beta} \overline{dx}^\alpha \overline{dx}^\beta + 2c dT_f \eta_{\alpha\beta} \zeta^\alpha \overline{dx}^\beta + c^2 dT_f^2 \eta_{\alpha\beta} \zeta^\alpha \zeta^\beta = 0, \quad (60)$$

where the 4-vector \overline{dx}^α is orthogonal to ζ^α , i.e.

$$\left[\frac{\eta_{\alpha\beta} \zeta^\beta}{(\eta_{\alpha\beta} \zeta^\alpha \zeta^\beta)^{1/2}} + f_\alpha \right] \overline{dx}^\alpha = 0. \quad (61)$$

For this reason the preceding equation takes the form

$$\eta_{\alpha\beta} v_{ph}^\alpha v_{ph}^\beta - 2c f_\alpha v_{ph}^\alpha (\eta_{\alpha\beta} \zeta^\alpha \zeta^\beta)^{1/2} + c^2 \eta_{\alpha\beta} \zeta^\alpha \zeta^\beta = 0, \quad (62)$$

where $v_{ph}^\alpha = \overline{dx}^\alpha / dT_f$ is a photon velocity as measured by an observer in the NIFR.

The covariant equation (62) become more clearly in the coordinate system where the conditions (55) are satisfied.

Indeed, it follows from the equality $\mathcal{F}(x, \zeta) = 1$ and from (55) and (58) that the quantity $(\eta_{\alpha\beta} \zeta^\alpha \zeta^\beta)^{1/2} = 1 - f_\alpha \zeta^\alpha = 1 - f_2 \zeta^2 - f_0 \zeta^0 = 1$ with accuracy up to $\Omega r/c$. Therefore, photon's velocity v_{ph} in the NIFR satisfy the equation

$$v_{ph}^2 + 2c v_{ph} f_i k^i - c^2 = 0, \quad (63)$$

where

$$v_{ph}^2 = -\eta_{ik} \frac{dx^i}{dT_f} \frac{dx^k}{dT_f}. \quad (64)$$

Thus, the velocity of an outgoing photon is given up to accuracy $\Omega r/c$ by the equation

$$v_{ph} = c(1 - f_i k^i). \quad (65)$$

The time of the photon motion around of the disk of the radius R is an integral of

$$dT_f = \frac{dL_f}{v_{ph}} = \frac{dL}{c} (1 + 2f_i k^i), \quad (66)$$

which yields for clockwise and counterclockwise directions $dT_{F+} = (dL/c)(1 + 2f_\theta)$ and $dT_{F-} = (dL/c)(1 - 2f_\theta)$, respectively. The difference in the time interval between light propagation on the rotating disk in two directions is $4\pi R^2 \Omega / c^2$, which gives the Signac phase shift [27]. Thus, the Signac effect for an isolated observer in the rotating frame can be treated as caused by the Finslerian metric of space-time in non-inertial frames of reference, see also [29].

6 Inertial Forces

Let us show that the existence of inertial forces in NIFRs can be interpreted as a manifestation of a Finslerian connection of space-time in these frames.

According to our main assumption in Sec. 2, the differential equations of the motion of the point masses in an IFR, forming the reference body of a NIFR, are equations of geodesic lines of space-time in this NIFR. These equations can be found from the variational principle $\delta \int ds = 0$ and are of the form

$$d\zeta^\alpha / ds + G^\alpha(x, \zeta) = 0, \quad (67)$$

where ζ^α is the 4-velocity of the point mass, the world line of which is $x^\alpha = x^\alpha(s)$, and

$$G^\alpha(x, \zeta) = \Gamma_{\beta\gamma}^\alpha \zeta^\beta \zeta^\gamma + \mathcal{B}_\beta^\alpha \zeta^\beta + \zeta^\alpha Z d(Z^{-1})/ds. \quad (68)$$

In these equations $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols in Minkowski space-time,

$$Z^{-1} = d\sigma/ds = (\eta_{\alpha\beta} \zeta^\alpha \zeta^\beta)^{1/2}, \quad \mathcal{B}_\beta^\alpha = \eta^{\alpha\delta} \mathcal{B}_{\delta\beta}$$

and

$$\mathcal{B}_{\delta\beta} = \partial f_{\beta}/\partial x^{\delta} - \partial f_{\delta}/\partial x^{\beta}.$$

In Finslerian space-time a number of connections compatible with (67) can be defined [23]. In particular, this equation can be interpreted in the sense that in NIFRs space-time the absolute derivative of a vector field $\xi^{\alpha}(x)$ along the world line $x^{\alpha} = x^{\alpha}(s)$ is given by

$$D\xi^{\alpha}/ds = d\xi^{\alpha}/ds + G_{\beta}^{\alpha}(x, dx/ds)\xi^{\beta}, \quad (69)$$

where

$$G_{\beta}^{\alpha}(x, dx/ds) = \mathcal{B}_{\beta}^{\alpha} + \Gamma_{\beta\gamma}^{\alpha} dx^{\gamma}/ds + Z dZ^{-1}/ds.$$

Equations (69) define a connection of the Laugwitz type [23] in the space-time of a NIFR, which is nonlinear with respect to dx^{α} . The change in the vector ξ^{α} due to an infinitesimal parallel transport is

$$d\xi^{\alpha} = -G_{\beta}^{\alpha}(x, dx)\xi^{\beta}, \quad (70)$$

Consider a free motion of a particle of mass m in a NIFR. Since $ds = d\sigma - f_{\alpha}dx^{\alpha}$, equations of the motion are

$$Du^{\alpha}/ds = \mathcal{B}_{\beta}^{\alpha}u^{\beta}. \quad (71)$$

The equations of the motion (67) on a non-relativistic disk rotating in the plane xy about the axis z with an angular velocity Ω are take the form

$$d\vec{\zeta}/dt + \vec{\Omega} \times \vec{r} = 0, \quad (72)$$

where $\vec{r} = \{x, y, z\}$ and the coordinates origin coincides with the disk center. The absolute derivative (69) of a vector $\vec{\xi}$ is given by

$$D\vec{\xi}/dt = d\vec{\xi}/dt - \vec{\Omega} \times \vec{\xi}. \quad (73)$$

and equations of the motion (71) of the considered particle in the NIFR are

$$D\vec{v}/dt = -\vec{\Omega} \times \vec{v}, \quad (74)$$

where $\vec{v} = \{\dot{x}, \dot{y}, \dot{z}\}$.

Next, for the 4-velocity u^{α} we have

$$u^{\alpha} = \bar{u}^{\alpha} + \zeta^{\alpha}, \quad (75)$$

where \bar{u}^{α} is the spatial velocity of the particle in the NIFR. In the non-relativistic limit (75) can be written in the form

$$\vec{v} = \vec{\bar{v}} + \vec{\zeta}, \quad (76)$$

where $\vec{\bar{v}}$ is the “relative velocity“ of the particle and $\vec{\zeta}$ is the velocities field of the points of the disk in the laboratory frame. Substituting (76) for (74), we find that

$$D\vec{\bar{v}}/dt = -D\vec{\zeta}/dt - \vec{\Omega} \times \vec{v} - \vec{\Omega} \times \vec{\zeta}. \quad (77)$$

The value $D\vec{\bar{v}}/dt$ is an acceleration of the considered particle as measured by an observer in the NIFR. The velocities field $\vec{\zeta}$ of the disk points is given by

$$\vec{\zeta} = \vec{\Omega} \times \vec{r}. \quad (78)$$

Hence, along the particle path we have $d\vec{\zeta}/dt = \vec{\Omega} \times \vec{v}$ and

$$D\vec{\zeta}/dt = d\vec{\zeta}/dt - \vec{\Omega} \times \vec{\zeta} = \vec{\Omega} \times \vec{v}. \quad (79)$$

Therefore, finally, we find from (74)

$$mD\vec{v}/dt = -2m(\vec{\Omega} \times \vec{v}) - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}). \quad (80)$$

We have arrived at non-relativistic equations of motion of a particle in rotating frames [30]. The right-hand side of (80) is the expression for the Coriolis and centrifugal forces.

Thus, in non-relativistic limit the Finslerian space-time in NIFRs manifests itself in the structure of the derivative of vectors with respect to time t , see also [29].

It is interesting to note that (73) is considered sometimes in classical dynamics nominally [30] just to obtain the expression for inertial forces in NIFRs.

7 Relativity of Inertia

A clock, which is in a NIFR at rest, is unaffected by acceleration in space-time of this frame. A difference in the rate of an ideal clock in IFRs and NIFRs is a real consequence of a difference between the space-time metric in the IFR and NIFR. It is given by the factor $Z = ds/d\sigma$. For a rotating disk of the radius R the factor $Z = 1 - \omega^2 R^2/2c^2$ which gives rise to the observed redshift in the well-known Pound-Rebka-Snyder experiments.

Consider another experimentally verifiable consequence of the above theory. Let $p^\alpha = mc \, dx^\alpha/d\sigma$ be 4-momentum of a particle in an IFR. From the point of view of an observer in a NIFR

$$p^\alpha = \bar{p}^\alpha + c^{-1} E \zeta^\alpha. \quad (81)$$

In this equations the spatial projection \bar{p}^α should be identified with the momentum and the quantity E with the energy of the particle.

It is obvious that $E = \zeta_\alpha^* p^\alpha$. Therefore, the energy of the particle in the NIFR is

$$E = m Z c^2 \zeta_\alpha^* u^\alpha, \quad (82)$$

where $Z = ds/d\sigma = F(x, dx/d\sigma)$. For the particle at rest in the NIFR $u^\alpha = \zeta^\alpha$ and we obtain

$$E = m Z c^2. \quad (83)$$

Thus, the inertial mass m_1 of the particle in the NIFR is given by

$$m_1 = Z m. \quad (84)$$

The quantity m coincides with the proportionality factor between the momentum and the velocity of a nonrelativistic particle in the NIFR.

Since Z is the function of x^α , the inertial mass m_n in the NIFR is not a constant. For example, on a rotating disk we have

$$m_n = m / (1 - \Omega^2 r^2 / 2c^2), \quad (85)$$

where Ω is the angular velocity and r is the distance of the body from the disk centre. The difference between the inertial mass m^{eq} of a body on the Earth equator and the mass m^{pol} of the same body on the pole is given by

$$(m^{eq} - m^{pol})/m^{pol} = 1.2 \cdot 10^{-12}. \quad (86)$$

The dependence of the inertial mass of particles on the Earth longitude can be observed by the Mössbauer effect. Indeed, the change $\Delta\lambda$ in a wave length λ at the Compton scattering on particles of the masses m is proportional to m . If this value is measured for gamma-quanta with the help of the Mössbauer effect at a fixed scattering angle, then after transporting the measuring device from the longitude φ_1 to the longitude φ_2 we obtain

$$\frac{\Delta\lambda_{\varphi_1}^{-1} - \Delta\lambda_{\varphi_2}^{-1}}{\Delta\lambda_{\varphi_1}^{-1}} = \Theta [\cos^2 \varphi_1 - \cos^2 \varphi_2], \quad (87)$$

where $\Theta = 1.2 \cdot 10^{-12}$.

8 Gravitation in Inertial and Proper Reference Frames

To implement the the bimetric description of gravity based on the relativity of space-time to the employed reference frame, suppose [7] that in Minkowski space-time gravitation can be described as a tensor field $\psi_{\alpha\beta}(x)$ of spin 2, for which the Lagrangian, describing the motion of a test particle of mass m is of the form

$$L = -mc[g_{\alpha\beta}(\psi) \dot{x}^\alpha \dot{x}^\beta]^{1/2}, \quad (88)$$

where $\dot{x}^\alpha = dx^\alpha/dt$ and $g_{\alpha\beta}$ is the symmetric tensor whose components are functions of $\psi_{\alpha\beta}$.

Consider a frame of reference, the reference body of which is formed by identical point masses m moving in an IFR under the effect of the field $\psi_{\alpha\beta}(x)$. It is a proper frame of reference of the given field. An observer, located in the PFR at rest, moves along a geodesic line of his space-time.

According to (33) the square of the line element in a PFR of the field $\psi_{\alpha\beta}$ is given by

$$ds^2 = g_{\alpha\beta}(\psi) dx^\alpha dx^\beta. \quad (89)$$

Thus, according to what has been outlined in Sec. 4, space-time in PFRs is Riemannian V_4 with the curvature other than zero. Viewed by an observer located in the IFR, the motion of the particles, forming the reference body of the PFR, is affected by the force field $\psi_{\alpha\beta}$. Let $x^i(t, \chi)$ be a set of paths of the motion of the depending on the parameter χ . Then, for the observer located in the IFR the relative motion of a pair of particles from the set is described in non-relativistic limit by the differential equations [32]

$$\frac{\partial^2 n^i}{\partial t^2} + \frac{\partial^2 U}{\partial x^i \partial x^k} n^k = 0, \quad (90)$$

where $n^k = \partial x^k / \partial \chi$ and U is the gravitational potential.

However, the observer in a PFR of this field will not feel the existence of the field since he moves in the space-time of the PFR along a geodesic line. The presence of the field $\psi_{\alpha\beta}$ will be displayed for him differently — as space-time curvature which manifests itself as a deviation of the world lines of nearby points of the reference body.

For a quantitative description of this fact it is natural for him to use the Riemannian normal coordinates.¹⁰ In these coordinates spatial components of the deviation equations of geodesic lines are

$$\frac{\partial n^i}{\partial t^2} + R_{0k0}^i n^k = 0, \quad (91)$$

where R_{0k0}^i are the components of the Riemann tensor. If the metric tensor does not depend on time t , then

$$R_{0k0}^i = -g^{ri} \frac{\partial^2 g_{00}}{\partial x^r \partial x^k}. \quad (92)$$

¹⁰ This and the above consideration does not depend on the used coordinate system, it can be performed by a covariant method.

In the Newtonian limit these equations coincide with (90).

Thus, in two frames of reference being used we have two different descriptions of particles motion — as moving under the action of a force field in Mankowski space-time, and as moving along the geodesic line in a Riemann space-time with the curvature other than zero.

Of course, (33) refers to any classical field. For instance, space-time in PFRs of an electromagnetic field is Finslerian. However, since ds , in this case, depends on the mass and charge of the particles forming the reference body, this fact is not of great significance.

If we start from the Lagrangian (88) for the motion of test particles, Einstein's equations cannot be considered as equations for finding functions $g_{\alpha\beta}(x)$. The reason is that the gravitational equations for $g_{\alpha\beta}(\psi)$ should be some differential equations which are invariant under a certain group of gauge transformations $\psi_{\alpha\beta}(x) \rightarrow \bar{\psi}_{\alpha\beta}(x)$ which are a consequence of the existence of "extra" components of the tensor $\psi_{\alpha\beta}$. These transformations induce some transformations $g_{\alpha\beta}(x) \rightarrow \bar{g}_{\alpha\beta}(x)$. Equations of motion of particles resulting from the Lagrangian (88) certainly should be invariant under such transformations of the tensor $g_{\alpha\beta}(x)$. But the equations of motion are, at the same times, differential equations of geodesic lines in Riemannian's space-time of a proper frame of reference. Consequently, if equations of gravitation do not contain functions $\psi_{\alpha\beta}$ explicitly, i.e. are differential equations for $g_{\alpha\beta}(x)$, they should be invariant under such transformations of metric tensor of space-time in PRFs, and it takes place in any coordinate system at that.

9 Spherically-Symmetric Gravitational Field

Let us find a spherically-symmetric vacuum solution $g_{\alpha\beta}(x)$ of bimetric equations (30) in Minkowski space-time E_4 . Such a solution, considered as a tensor field in E_4 , describes, in particular, the gravity of a point mass M for a remote observer in an IFR for which space-time is supposed to be pseudo-Euclidean. In this case, if the Lagrangian of a particle is invariant under the mapping $t \rightarrow -t$, in the spherical coordinates of the Minkowski space-time, it is of the form

$$L = -m c [A\dot{r}^2 + B(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) - c^2 C]^{1/2}, \quad (93)$$

where A , B and C are the functions of the radial coordinate r .

The associated line element of space-time in PFRs is

$$ds^2 = A dr^2 + B[d\theta^2 + \sin^2 \theta \varphi^2] - C dx^{02}, \quad (94)$$

Some additional conditions can be imposed on the tensor $g_{\alpha\beta}$ because of the gauge (geodesic) invariance. In particular, under the conditions $Q_\alpha = 0$ Eqs. (22) are reduced to Einstein's vacuum equations $R_{\alpha\beta} = 0$. Thus, the functions A , B and C can be found as a solution of the system of the differential equations :

$$R_{\alpha\beta} = 0 \quad (95)$$

and

$$Q_\alpha = 0, \quad (96)$$

which satisfy the conditions:

$$\lim_{r \rightarrow \infty} A = 1, \quad \lim_{r \rightarrow \infty} (B/r^2) = 1, \quad \lim_{r \rightarrow \infty} C = 1. \quad (97)$$

It allows to use all mathematical methods elaborated for general relativity.

Conditions (96) yield one equation:

$$B^2 AC = r^4. \quad (98)$$

It allows to exclude the function A from the (95). Then the equations $R_{11} = 0$ and $R_{00} = 0$ are:

$$-2BC' + 2rB'C' + rBC'' = 0, \quad (99)$$

$$-4BCB' + rCB'^2 - 2BC' + 2rBB'C' + 2rBCB'' + rB^2C'' = 0. \quad (100)$$

Because of equality (99) the sum of tree terms in (100) is equal to zero, and we obtain a differential equation for the function B :

$$2rBB'' + rB'^2 - 4BB' = 0. \quad (101)$$

A general solution of this equation can be written as $B = a(r^3 + \mathcal{K}^3)^{2/3}$ where a and \mathcal{K} are some constants. The constant $a = 1$ which can be seen from condition (97).

The function C can be found from differential equation (99) in the form

$$C'' + 2\frac{rB' - B}{rB}C' = 0, \quad (102)$$

where $(rB' - B)/rB = (r^3 - \mathcal{K}^3)/(r^4 + r\mathcal{K}^3)$. A general solution of this equation is $C = b - \mathcal{Q}/B^{1/2}$ where b and \mathcal{Q} are constant. It follows from (97) that $b = 1$.

Thus, in the spherical coordinate system the functions A , B and C are given by:

$$A = \frac{r^4}{f^4(1 - \mathcal{Q}/f)}, \quad B = f^2, \quad C = 1 - \frac{\mathcal{Q}}{f}, \quad (103)$$

where

$$f = (r^3 + \mathcal{K}^3)^{1/3}. \quad (104)$$

The nonzero components of the tensor $B_{\alpha\beta}^\gamma$ are given by

$$\begin{aligned} B_{rr}^r &= \frac{1}{2} \frac{A'}{A}; & B_{\theta\theta}^r &= \frac{1}{2} \frac{2rA - B'}{A}; & B_{\phi\phi}^r &= -\frac{1}{2} \frac{2rA - B'}{A} \sin^2 \theta; \\ B_{tt}^r &= \frac{1}{2} \frac{C'}{A}; & B_{tr}^t &= \frac{1}{2} \frac{C'}{C}; & B_{\theta r}^\theta &= \frac{1}{2} \frac{2B - B'r}{rB}; & B_{\phi r}^\phi &= \frac{2B - B'r}{rB}. \end{aligned}$$

In non-relativistic limit the radial component of the equations of the motion of a test particle (1) takes the form $\ddot{x}^r = -c^2 \Gamma_{00}^r$, where $\Gamma_{00}^r = C'/2A = r^4 C'/f^4 C$. Therefore, to obtain the Newton gravity law it should be supposed that at large r the function $f(r)$ coincides with r , and $\mathcal{Q} = r_g = 2GM/c^2$ is the classical Schwarzschild radius.

At the given constant \mathcal{Q} allowable solutions (103) are obtained by changing the arbitrary constant \mathcal{K} . In particular, if we set $\mathcal{K} = 0$, then the line element (94) of the space-time in PFRs coincides with the Droste-Weyl solution of Einstein's equations [34] (it is commonly named Schwarzschild's solution) which has an event horizon at $r = r_g$:

$$ds^2 = -\frac{dr^2}{(1 - r_g/r)} - r^2[d\theta^2 + \sin^2 \theta d\varphi^2] + (1 - r_g/r) dx^{02}. \quad (105)$$

If we set $\mathcal{K} = \mathcal{Q}$, the line element coincides with the original Schwarzschild solution [35]

$$ds^2 = -\frac{f'^2 dr^2}{(1 - r_g/f)} - f^2[d\theta^2 + \sin^2 \theta d\varphi^2] + (1 - r_g/f) dx^{02}, \quad (106)$$

where $f = (r_g^3 + r^3)^{1/3}$. This solution has no event horizon and no singularity in the centre. Really, for motion of a particle in the plane $\theta = \pi/2$ laws of conservation of energy E and momentum J take place:

$$\dot{r} \frac{\partial L}{\partial \dot{r}} - \dot{\varphi} \frac{\partial L}{\partial \dot{\varphi}} - L = E, \quad \frac{\partial L}{\partial \dot{\varphi}} = J. \quad (107)$$

It follows from this fact that the equations of the motion are of the form

$$\dot{r}^2 = (c^2 C/A)[1 - (C/\bar{E}^2)(1 + r_g^2 \bar{J}^2/B)], \quad (108)$$

$$\dot{\varphi} = c C \bar{J} / r_g B \bar{E}, \quad (109)$$

where (t, r, φ) are the spherical coordinates, $\dot{r} = dr/dt$, $\dot{\varphi} = d\varphi/dt$, $\bar{E} = E/mc^2$, $\bar{J} = J/r_g mc$.

The radial velocity of freely falling particle is given by the equation

$$v = c \left[\frac{C}{A} (1 - C) \right]^{1/2} \quad (110)$$

Fig. 1 shows the plot of the velocity as a function of the distance r/r_g from the centre of a point attracting mass. It follows from this figure that this function is defined at all distances from the centre, and tends to zero at $r \rightarrow 0$.

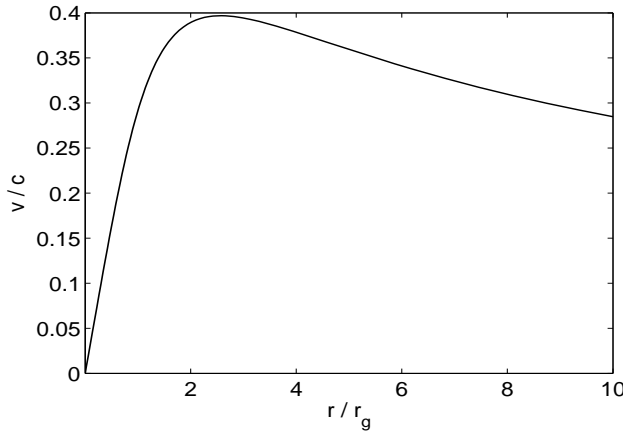


Fig. 1 The velocity of a freely falling particle (in units of c) as the function of the distance r/r_g from the centre.

Line elements (105) and (106) in V_4 are not equivalent because they were obtained in the same coordinate system and correspond to the different values of the constant \mathcal{K} in solution (103). Both of the line elements refer to the spherical coordinate system, defined in E_4 , where t, r, φ and θ are magnitudes measured by measurement instruments. Suppose, as it is usually believed, (106) can be transformed to the form (105) by an appropriate local coordinate transformation $r \rightarrow r'$ and $t \rightarrow t'$.¹³ and [37]. But these new coordinates have not an operational meaning, and after an appropriate transformation of the physical 3-space and time intervals, the same physical results will be obtained from (105) as from (106). (Just as in classical electrodynamics in arbitrary coordinates). In particular, since the line element (106) at $\mathcal{K} = \mathcal{Q}$ has no the event horizon and the singularity in the centre in spherical coordinates, it does not contain it in other coordinate systems.

¹³ Problems of correctness of such transformation have been considered in [34], and criticism with respect to the Droste-weyl solution in General Relativity — in [36]

It can be argued that the constant \mathcal{K} must be equal to $\mathcal{Q} = r_g$. Gauge-invariant tensor $B_{\beta\gamma}^\alpha(x)$ can be considered as the field strength tensor of gravitational field. There is a simple possibility to consider the object $B_{\beta\gamma}^\alpha(x)$ as a function of a tensor field $\psi_{\alpha\beta}(x)$ which may be interpreted as a potential $\psi_{\alpha\beta}(x)$ described gravity in E_4 as a field of spin two. Namely, one can set

$$B_{\beta\gamma}^\alpha = \nabla^\alpha \psi_{\beta\gamma} - (n+1)^{-1} (\delta_\beta^\alpha \nabla^\sigma \psi_{\sigma\gamma} + \delta_\gamma^\alpha \nabla^\sigma \psi_{\beta\sigma}).$$

The identity $B_{\alpha\gamma}^\gamma = 0$ is satisfied as it is to be expected according to the definition of the tensor $B_{\alpha\gamma}^\gamma$. Then, at the gauge condition $\nabla^\sigma \psi_{\sigma\gamma} = 0$ Eqs. (22) are

$$\square \psi_{\alpha\beta} = \nabla^\sigma \psi_{\alpha\gamma} \nabla^\gamma \psi_{\sigma\beta}, \quad \nabla^\sigma \psi_{\sigma\gamma} = 0, \quad (111)$$

where \square is the covariant D'Alambert operator in Minkowski space-time.

It is natural to suppose that in the presence of matter these equations take the form¹⁴

$$\begin{aligned} \square \psi_{\alpha\beta} &= \kappa (T_{\alpha\beta} + t_{\alpha\beta}), \\ \nabla^\sigma \psi_{\sigma\gamma} &= 0, \end{aligned} \quad (112)$$

where $\kappa = 8\pi G/c^4$,

$$t_{\alpha\beta} = \kappa^{-1} B_{\alpha\mu}^\nu B_{\beta\nu}^\mu = \kappa^{-1} \nabla^\sigma \psi_{\alpha\gamma} \nabla^\gamma \psi_{\sigma\beta}, \quad (113)$$

and $T_{\alpha\beta}$ is the energy-momentum tensor of matter.

Obviously, the equality

$$\nabla^\beta (T_{\alpha\beta} + t_{\alpha\beta}) = 0 \quad (114)$$

is valid. Therefore, the magnitude $t_{\alpha\beta}$ can be interpreted as the energy-momentum tensor of the gravitational field.

Let us find the energy of the gravitational field of a point mass M as an integral in Minkowski space-time:

$$\mathcal{E}_{gf} = \int t_{00} dV, \quad (115)$$

where dV is a volume element. In Newtonian theory this integral is divergent. In our case we have:

$$t_{00} = 2\kappa^{-1} B_{00}^1 B_{01}^0 = \frac{c^4}{8\pi G} \frac{\mathcal{Q}^2}{f^4 \mathcal{K}} \quad (116)$$

and, therefore, in spherical coordinates, we obtain

$$\mathcal{E}_{gf} = \int t_{00} dV = \frac{1}{8} \frac{\mathcal{Q}^2 c^4}{\pi G} \mathcal{J}, \quad (117)$$

where [38]

$$\mathcal{J} = \int \frac{dV}{f^4} = \frac{4\pi}{3\mathcal{K}} B(1, 1/3) \quad (118)$$

and

$$B(z, w) = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} dt \quad (119)$$

¹⁴ It should be noted that, when we introduce $\psi_{\alpha\beta}$ in such a way, we cannot be sure a priori that the equations for $\psi_{\alpha\beta}$ yield all physical solutions of the equations for $B_{\beta\gamma}^\alpha$.

is the B-function. Using the equality

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad (120)$$

where Γ is the Γ -function, we obtain $\mathcal{J} = 4\pi/\mathcal{K}$, and, therefore,

$$\mathcal{E}_{gf} = \frac{\mathcal{Q}}{\mathcal{K}} \frac{\mathcal{Q}c^4}{2G} = \frac{\mathcal{Q}}{\mathcal{K}} M c^2. \quad (121)$$

We arrive at the conclusion that at $\mathcal{K} \neq 0$ the energy of a point-mass is finite and, in particular, at $\mathcal{K} = \mathcal{Q}$ it is caused entirely by its gravitational field:

$$\mathcal{E}_{gf} = M c^2. \quad (122)$$

The spatial components of the vector $P_\alpha = t_{0\alpha}$ are equal to zero.

Due to these facts we assume in the present paper that $\mathcal{K} = \mathcal{Q} = r_g$ and consider the functions

$$\begin{aligned} A &= \frac{r^4}{f^4(1 - \mathcal{Q}/f)}, \quad C = 1 - r_g/f, \quad B = f^2; \\ f &= (r_g^3 + r^3)^{1/3} \end{aligned} \quad (123)$$

in the Lagrangian (93) as a basis for analysis of physical effects.

It must be noted that by using the above results we can find also the gravitational field inside a spherically-symmetric matter layer. In order to reach a coincidence of the equation of motion of test particles in nonrelativistic limit with the Newtonian one, the constant \mathcal{Q} in (103) in this case must be equal to zero. Therefore, the spherically-symmetric matter layer does not create gravitational field inside itself. This result will be used in next section.

As an observer, located in a PFR does not feel influence of any forces, he should explain the fact of the deviation of paths of close particles of the PFR reference body as a manifestation of deviation of geodesic lines of these point masses in space-time with the line element (106). However, unlike the situation in General Relativity, he does not observe any problems at the Schwarzschild radius and close to $r = 0$ because components of the curvature tensor have not any singularity. For example $R_{010}^1 = r_g f^{-4}(r_g - f)$. It tends to zero when $r \rightarrow 0$. This fact is in accordance with the fact that in flat space-time the gravitational force tends to zero when the distance from the central point mass tends to zero, see next section.

10 Cosmological test

Physical consequences from the spherical-symmetric solution of equations (30) differ very little from those in general relativity at distances $r \gg r_g$ but they are different in principle at distances of the order of the Schwarzschild radius or less than that. Some principal physical consequences were considered in papers [40], [41], [42]. Some of them have chances to be checked up. The equations (30) were successfully tested by the binary pulsar *PSR1913 + 16* [39]. Here we consider another problem.

Within last 8 years numerous data were obtained testifying that the most distant galaxies move away from us with acceleration [43]. This fact poses serious problems both for astrophysics and for fundamental physics [44]. In the present paper it is shown that available observed data are an inevitable consequence of properties of the gravitational force as deduced from the geodesic-invariant bimetric gravitation equations.

According to (1), the gravitational force $m \ddot{r}$ of a point mass M affecting a freely falling particle of mass m in Minkowski space-times is given by

$$F = -m [c^2 C' / 2A + (A' / 2A - 2C' / 2C) \dot{r}^2], \quad (124)$$

where the functions A and C are given by Eqs. (123) and r is the distance from the centre.

For particles at rest ($\dot{r} = 0$)

$$F = -\frac{GmM}{r^2} \left[1 - \frac{r_g}{(r^3 + r_g^3)^{1/3}} \right]. \quad (125)$$

Fig. 2 shows the force F affecting particles at rest and the ones, free falling from infinity on a point mass, as the function of the distance $\bar{r} = r/r_g$ from the centre.

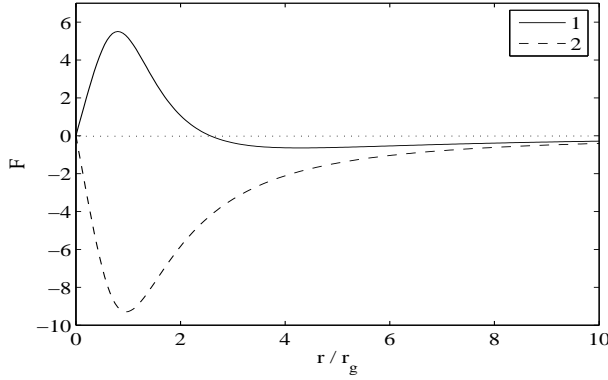


Fig. 2 The gravitational force (arbitrary units) affecting freely particles (curve 1) and particles at rest (curve 2) near an attractive point mass.

It follows from this figure that the main peculiarity of the force acting on particles at rest in Minkowski space-time is that it tends to zero at $r \rightarrow 0$. The gravitational force acting on freely moving particles essentially differs from that acting on particles at rest. In this case the force changes its sign at $\sim 2r_g$ and becomes repulsive. Although we have so far never observed the motion of particles at distances of the order of r_g , we can verify this result for very remote objects in the Universe at large cosmological redshifts, because it is well-known that the radius of the observed region of the Universe is of the order of its Schwarzschild radius.

A magnitude which is related to observations of the expanding Universe is a velocity of distant star objects moving off from the observer. To find the radial velocity of such objects which is at the distance R from the observer, we can use spherically-symmetric solution (123) of the vacuum equations (22) taking into account the effect of gravity of a spherically symmetric matter layer, as pointed out at the end of Sec. 8. In this case $M = (4/3)\pi\rho R^3$ is the matter mass inside the sphere of the radius R , ρ is the observed matter density, and $r_g = (8/3)\pi c^{-2}G\rho R^3$ is Schwarzschild's radius of the matter inside of the sphere.

There is no necessity to demand the global spherical symmetry of matter outside of the sphere. Indeed, suppose, the matter density ρ at the moment is of the order of the observed density of the matter in the Universe ($10^{-29} - 10^{-30} \text{ g cm}^{-3}$). Fig. 3 shows the radial acceleration \ddot{R} of a particle in the expanding Universe on the surface of the sphere of the radius R .

Two conclusions can be drawn from this figure.

1. At some distance from the observer the relative acceleration changes its sign. If the $R < 2 \cdot 10^{27} \text{ cm}$, radial acceleration of particles is negative. If $R > 2 \cdot 10^{27} \text{ cm}$, its magnitude is positive. Hence, for sufficiently large distances the gravitational force affecting particles is repulsive and gives rise to a relative acceleration of particles.
2. The gravitational force, affecting the particles, tends to zero when R tends to infinity. The same fact takes place as regards the force acting on particles in the case of a static matter. The reason of the fact is that at a sufficiently large distance R from the observer the Schwarzschild radius of the matter inside the sphere of radius R becomes of the same order as its radius. Approximately at $R \sim 2r_g$ the gravitational force begins

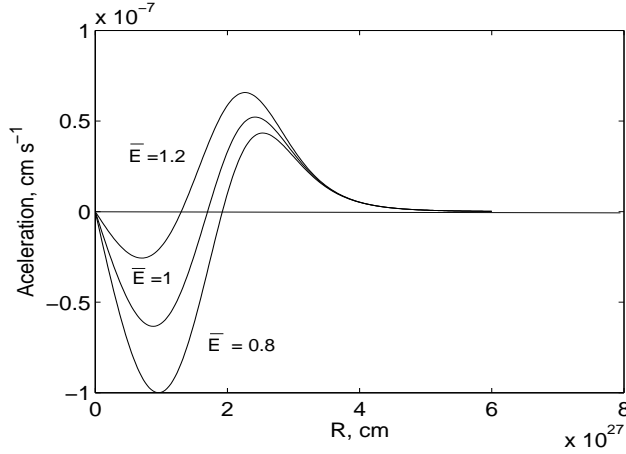


Fig. 3 The acceleration of particles on the surface of a homogeneous sphere of radius R for three value of the parameter \bar{E} . The matter density is equal to $10^{-29} \text{ g cm}^{-3}$.

to decrease. The ratio R/r_g tends to zero when R tends to infinity, and under these circumstances the gravitational force in the theory under consideration tends to zero. Therefore, the gravitational influence of very distant matter is negligible small.

Thus, according to (108), the radial velocity of a star object at the distance R from the observer takes the form

$$v = c \frac{C f^2}{R^2} \sqrt{1 - \frac{C}{\bar{E}^2}}, \quad (126)$$

where C and f are functions of distance R of the object from an observer, $v = dR/dt$, \bar{E} is the total energy of a particle divided by mc^2 .

Proceeding from this equation we will find Hubble's diagram following mainly the method being used in [45]. Let ν_0 be a local frequency in the proper reference frame of a moving source at the distance R from the observer, ν_l be this frequency in a local inertial frame, and ν be the frequency as measured by the observer in the centre of the sphere. The redshift $z = (\nu_0 - \nu)/\nu$ is caused by both Doppler-effect and gravitational field. The Doppler-effect is a consequence of a difference between the local frequency of the source in inertial and comoving reference frame, and it is given by [22]

$$\nu_l = \nu_0 [1 - \sqrt{(1 - v/c)(1 + v/c)}]. \quad (127)$$

The gravitational redshift is caused by the matter inside the sphere of the radius R . It is a consequence of the energy conservation for photon. The energy integral (107) for the motion of free particles together with the Lagrangian (93) yield at $\dot{v} = 0$ and $\dot{\varphi} = 0$ the rest energy of a particle in gravitational field:

$$E = mc^2 \sqrt{C}. \quad (128)$$

Therefore, the difference in two local level energy E_1 and E_2 of an atom in the field is $\Delta E = (E_2 - E_1)\sqrt{C}$, so that the local frequency ν_0 at the distance R from an observer are related with the observed frequency ν by equality

$$\nu = \nu_l \sqrt{C}, \quad (129)$$

where we take into account that for the observer location $\sqrt{C} = 1$. It follows from (129) and (127) that the relationship between the frequency ν as measured by the observer and the proper frequency ν_0 of the

moving source in the gravitational field takes the form

$$\frac{\nu}{\nu_0} = \sqrt{C \frac{1 - v/c}{1 + v/c}}, \quad (130)$$

which in the newtonian limit is of the form

$$\nu = \left(1 - \frac{v}{c} - \frac{4}{3} \frac{\pi G \rho R^2}{c^2}\right) \nu_0. \quad (131)$$

Equation (130) yields the quantity $z = \nu_0/\nu - 1$ as a function of R . By solving this equation numerically we obtain the dependence $R = R(z)$ of the measured distance R as a function of the redshift. Therefore the distance modulus to a star object is given by

$$\mu = 5 \log_{10}[R(z)(z+1)] - 5. \quad (132)$$

where $R(1+z)$ is a bolometric distance (in pc) to the object.

If we demand that (126) has to give a correct radial velocity of distant star objects in the expansive Universe, it has to lead to the Hubble law at small distances R . At this condition the Schwarzschild radius $r_g = (8/3)\pi G \rho R^3$ of the matter inside the sphere is very small compared with R . For this reason $f \approx r$, and $C = 1 - r_g/r$. Therefore, at $\bar{E} = 1$, we obtain from (126) that

$$v = HR, \quad (133)$$

where

$$H = \sqrt{(8/3)\pi G \rho}. \quad (134)$$

If $\bar{E} \neq 1$ equation (126) does not lead to the Hubble law since v does not tend to 0 when $R \rightarrow 0$. For this

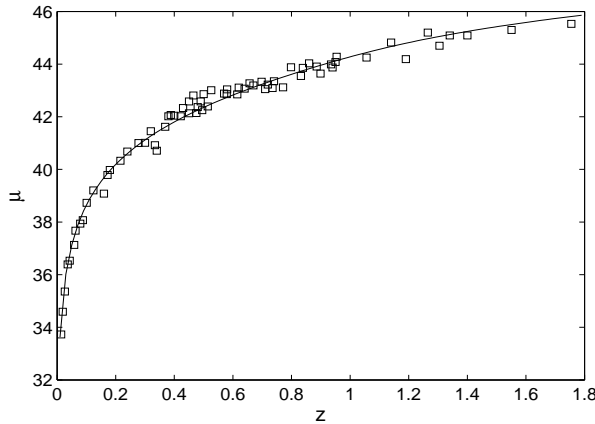


Fig. 4 The distance modulus μ vs. redshift z for the density $\rho = 4.5 \cdot 10^{-30} g cm^{-3}$. Small squares denote the observation data according to Riess et al.

reason we set $\bar{E} = 1$ and look for the value of the density at which a good accordance with observation data can be obtained. Fig. (4) show the Hubble diagram obtained by (132) compared with the Riess et al data [43]. It follows from this figure that the model under consideration does not contradict observation data. With the value of the density $\rho = 4.5 \cdot 10^{-30} g cm^{-3}$ we obtain from (134) that

$$H = 1.59 \cdot 10^{-18} c^{-1} = 49 (km/s) / Mpc. \quad (135)$$

Fig. 5 shows the dependence of the radial velocity v on the redshift. It follows from this figure that at $z > 1$ the Universe expands with an acceleration. At $R \rightarrow \infty$ the velocity and acceleration tend to zero.

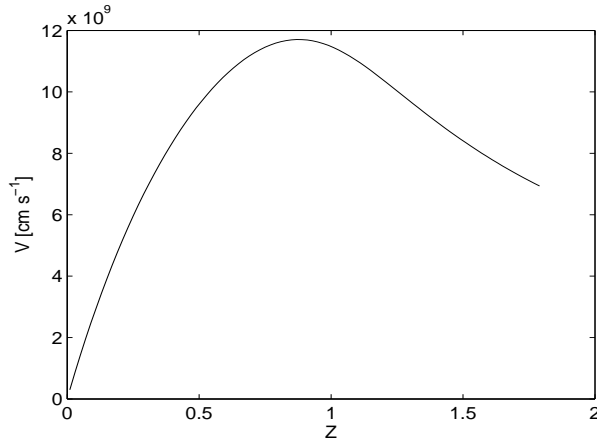


Fig. 5 The radial velocity vs. redshift z for the density $\rho = 4.5 \cdot 10^{-30} \text{ g cm}^{-3}$

11 Conclusion

The necessity of a discussion of the physical description of gravity through geodesic-invariant equations is quite obvious. However, the satisfactory implementation of such a programme is a difficult task. The equations of gravitation considered in the given paper do not contradict observation data, and their some predictions seem rather attractive. However it is still unclear whether they are only possible equations. Besides, it appears that the correct equations of this type should be the bimetric ones. However physical interpretation of such bimetricity involves deep problems of the space-time theory which presently have no a satisfactory understanding and a complete solution.

References

- [1] H. Weyl, Göttinger Nachr., 90 (1921).
- [2] T. Thomas, The differential invariants of generalized spaces, (Cambridge, Univ. Press) (1934).
- [3] L. Berwald, Ann. Math., **37**, 879 (1936)
- [4] L. Eisenhart, Riemannian geometry, (Princeton, Univ. Press) (1950).
- [5] A. Petrov, Einstein Spaces, (New-York-London, Pergamon Press. (1969).
- [6] N. Rosen, Gen. Relat. Grav., **4**, 435 (1973).
- [7] W. Thirring, Ann. Phys., **16**, 96 (1961).
- [8] S. Deser, Gen. Relat. Grav., **1**, 9 (1970).
- [9] V. Ogievetski and I. Polubariniv, Ann.Phys., **35**, 167 (1965).
- [10] Ya. Zeldovich and L. Grishchuk, Sov. Phys. Uspekhi, **31**, 666 (1988).
- [11] F. Hehl, J. McCrea, E. Mielke, Y. Ne'eman, Phys. Report, **258**, 1 (1995).
- [12] L. Verozub, L. Phys. Lett. A, **156**, 404 (1991).
- [13] G. Berkeley, Philosophical Works, M. Ayers, ed., (London, Dent) (1975).
- [14] Leibniz-Clarke Correspondence, H./, Alexander, ed, (Manchester University Press) (1956).
- [15] E. Mach, The Science of Mechanics, (Chicago, Open Court) (1915).
- [16] H. Poincaré, Dernières pensées, (Paris, Flammarion) (1913)
- [17] J. Monaghan, Ann. Rev. Astron. Astrophys., **30**, 543 (1992).
- [18] J. Monaghan and D. Rice, Month. Not. Royal Astron. Soc., **328**, 381 (2001).
- [19] L. Verozub, Ukr. Phys. Journ., **26**, 131 (1981) (In Russian).
- [20] L. Verozub, In: Recent Develop. in Theor. and Exp. Gen. Relat., Gravit. and Field Theor., T. Piran and R. Ruffini, eds, (World Sci. Pub.), 489 (1999).
- [21] L. Verozub, AIP Conf. Proc., **624**, 32 (2002).
- [22] L. Landau, and E. Lifshitz, The Classical Theory of Field, (Massachusetts, Addison - Wesley) (1971).
- [23] H. Rund, The differential geometry of Finsler space, (Berlin-Göttingen-Heidelberg, Springer), (1959).
- [24] L. Verozub, e-preprint gr-qc/0703083.

- [25] H. Dehnen, *Wissensch. Zeitschr. der Friedrich - Schiller Universitat, Jena, Math. - Naturw. Reihe. H.1 Jahrg.*, **15**, 15 (1966).
- [26] V. Antonov, V. Efremov, & Yu. Vladimirov, *Gen.Relat. Grav.*, **9**, 9 (1978).
- [27] E. Post, *Rev. Mod. Phys.* **39**, 475 (1967).
- [28] A. Ashtekar and A. Magnon, *Journ. of Math. Phys.*, **16**, 341 (1975).
- [29] L. Verozub, *Ukr. Phys. Journ.*, **26**, 778 (In Russian) (1981).
- [30] J. Syng, *Classical dynamics*, (Berlin-Göttingen-Heidelberg, Springer) (1960).
- [31] L. Verozub, *Ukr. Phys. Journ.*, **26**, 1598 (In Russian) (1981).
- [32] Ch. Misner, K. Thorne, J. Wheeler, *Gravitation*, (San Francisco, Feeman and Comp.) (1973).
- [33] H. Treder, *Gravitationstheorie und Äquivalenzprinzip*, (Berlin, Akademie-Verlag-Berlin) (1971).
- [34] L. Abrams, *Phys.Rev.*, **20**, 20 (1980).
- [35] K. Schwarzschild, *d. Berl. Akad.*, 189 (1916).
- [36] S. Crothers, *Progr. Phys.*, **1**, 68 (2005).
- [37] S. Antoci and D. Liebscher, e-print gr-qc/0102084 (2001).
- [38] P. Byrd and M. Friedman, *Handbook of elliptic integrals for engineers and physicists*, (Berlin-Göttingen-Heidelberg, Springer) (1954).
- [39] L. Verozub & A. Kochetov, *Grav and Cosmol.*, **6**, 246 (2000).
- [40] L. Verozub, *Astr. Nachr.*, **317**, 107 (1996).
- [41] L. Verozub, & A. Kochetov, *Astr. Nachr.*, **322**, 143 (2001).
- [42] L. Verozub, *Astr. Nachr.*, **327**, 355 (2006).
- [43] A. Riess et al, *Astr.Journ.*, **116**, 1009 (1998).
- [44] S. Weinberg, e-preprint astro-ph/0005265 (2000).
- [45] Ya. Zeldovich and I. Novikov, *Relativistic Astrophysics*, v.1, (New-York, Dover) (1996).
- [46] S. Weinberg, *Gravitation and Cosmology*, (New-York-London-Sydney-Toronto, J. Wiley & SonsInc.) (1972).